Decision Procedures for Flat Array Properties

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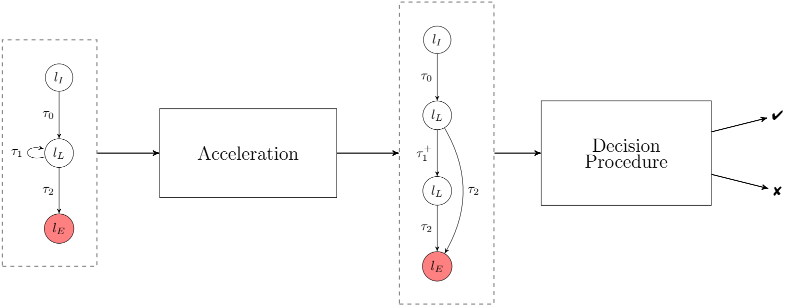
Context: quantified fragments of array theories

Many applications:

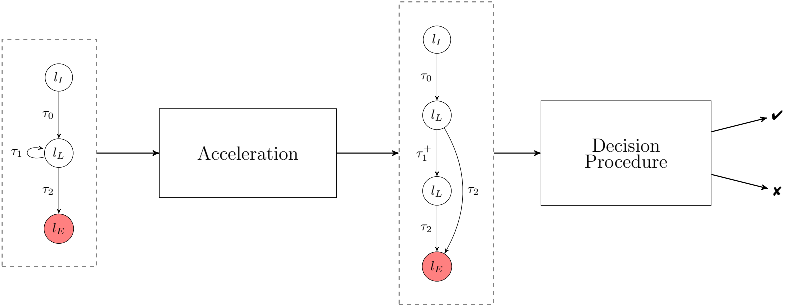
- Properties of the heap
- Frame axioms
- Checking user provided assertions
- Parameterized systems

$\Rightarrow$ Verifying array programs:
  - CEGAR-based approaches for array programs [AlbertiBG$^+$12]
  - **Accelerations of relations over arrays** [AlbertiGS13]

Accelerations of relations over arrays



Accelerations of relations over arrays



✔ Accelerations of a class of relation over arrays is definable via $\exists^*\forall^*$-formulæ [AlbertiGS13]

Accelerations might be outside known decidable fragments [BradleyMS06, HabermehlIV08, GeM09].

Accelerations of relations over arrays

$$\tau := G(i, \mathbf{a}[i]) \quad \wedge \quad i' = i + \bar{k} \quad \wedge \quad \mathbf{a}' = \mathsf{store}(\mathbf{a}, i, \mathbf{t}(\mathbf{a}[i]))$$

$$\Downarrow$$

$$\tau^+ := \exists y > 0. \begin{pmatrix} \forall j.\ [\ i \le j < i + \bar{k} \cdot y \wedge D_{\bar{k}}(j - i) \quad \to \quad G(\ j, \mathbf{a}(j)\ )\ ]\ \wedge \\ i' = i + \bar{k} \cdot y\ \wedge \\ \forall j.\ [\ \mathbf{a}'(j) = \mathbf{U}(\ i, j, y, \mathbf{a}(j)\ )\ ] \end{pmatrix}$$

Theory of arrays: "base" theory T + free functions $\mathbf{a}$

Fragment of interest: $\varphi := \exists \mathbf{c} \, \forall \mathbf{i} \; \psi(\; \mathbf{c} \; , \; \mathbf{i} \; , \; \mathbf{a}(t) \;)$

Quantified fragments of array theories

Related work

Theory of arrays: "base" theory T + free functions $\mathbf{a}$

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- In general, undecidable
- If constrained, two main strategies to show decidability:
  1. Instantiation-based
  2. Automata-based

Quantified fragments of array theories

Related work

Bradley et al. "What's decidable about arrays?", VMCAI 2006.

- Array property: $\varphi \ := \ \forall \mathbf{i}. F(\mathbf{i}) \rightarrow G(\ \mathbf{a}(\mathbf{i})\ )$
  - $F(\mathbf{i})$ is a conjunction of atoms of the kind $i \leq j$, $i \leq t$, $t \leq i$

I. Identify an *index set* $\mathcal{I}$
II. Instantiate $\mathbf{i}$ over $\mathcal{I}$ to obtain a quantifier-free $\psi_1 \wedge \cdots \wedge \psi_n$
III. Standard theory-combination approaches on $\psi_1 \wedge \cdots \wedge \psi_n$

- Complexity: NExpTime (NP if we fix the number of index variables)

Habermehl et al. “A Logic of Singly Indexed Arrays”, LPAR 2008.

- $\varphi \ := \ \forall \mathbf{i}. F(\mathbf{i}) \to G(\mathbf{i}, \mathbf{a}(\mathbf{i} + \bar{\mathbf{k}}))$
    - No disjunctions in G
    - Atoms are difference logic constraints (with equations modulo $\bar{\mathbf{k}}$)

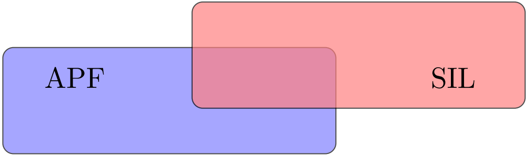
I. Translate φ into a FCADBM[1] $\mathcal{A}_\varphi$
II. Check the emptiness of $\mathcal{L}(\mathcal{A}_\varphi)$

- Complexity: unknown

[1] Deterministic flat counter automata with difference bound transition rules

Quantified fragments of array theories

Our contribution wrt related work



Quantified fragments of array theories

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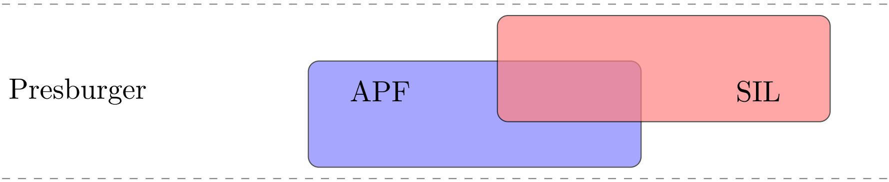

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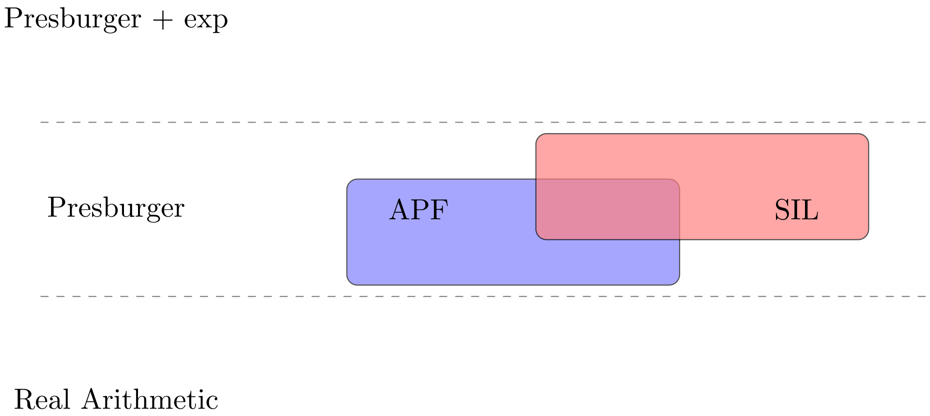

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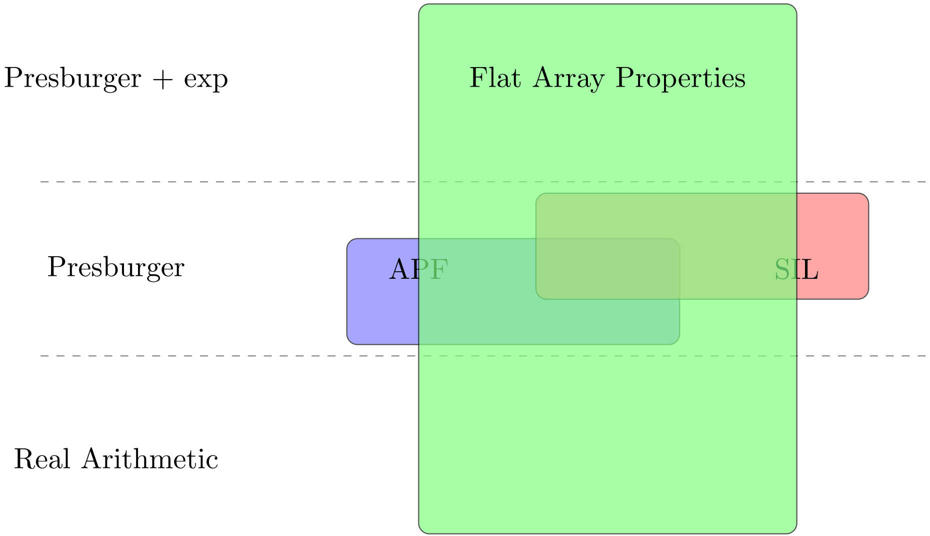

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Flat Array Properties

- $\varphi \ := \ \exists \mathbf{c}\, \forall \mathbf{i}. \psi(\ \mathbf{i} \ , \ \mathbf{a}(\mathbf{i}) \ , \ \mathbf{c} \ , \ \mathbf{a}(\mathbf{c}) \ )$
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- Mono-sorted theory: $T \cup \{a_1, \ldots, a_n\}$
  - $|\mathbf{i}| = 1$
  - Requirement: T-decidability of $\exists^*\forall\exists^*$-formulæ
  - Complexity: quadratic instance of a $\exists^*\forall\exists^*$ T-satisfiability problem

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- Multi-sorted theory: $T_I \cup T_E \cup \{a_1, \ldots, a_n\}$
  - `INDEX` atoms with at most one universally quantified variable
  - Requirement: T_I-decidability of $\exists^*\forall$-formulæ
  - Requirement: T_E-decidability of quantifier-free formulæ
  - Complexity if T_I, T_E are $\mathbb{P}^+$: NExpTime-complete

Decision Procedure for the multi-sorted case

$$F := \exists \mathbf{c}\ \forall \mathbf{i}\ .\psi(\ \mathbf{i}, a(\mathbf{i}), \mathbf{c}, a(\mathbf{c})\ )$$

$$\mathcal{M} \models F$$

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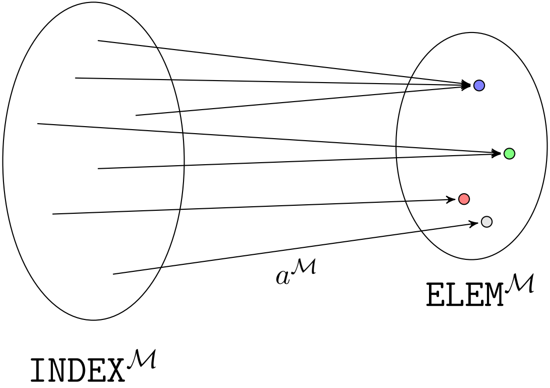

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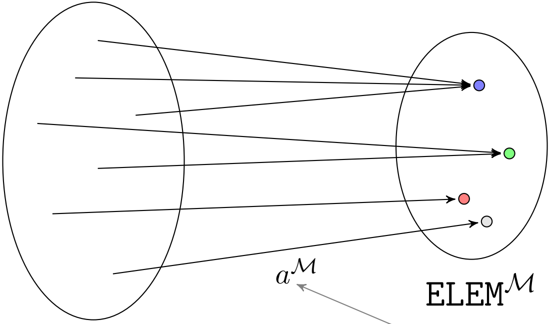

$\texttt{INDEX}^{\mathcal{M}}$

$a^{\mathcal{M}}$ is a *total* function from $\texttt{INDEX}^{\mathcal{M}}$ to $\texttt{ELEM}^{\mathcal{M}}$

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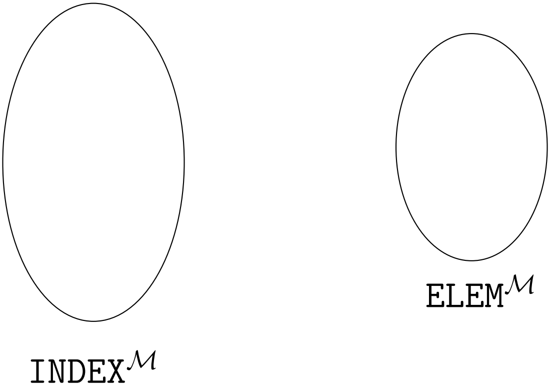

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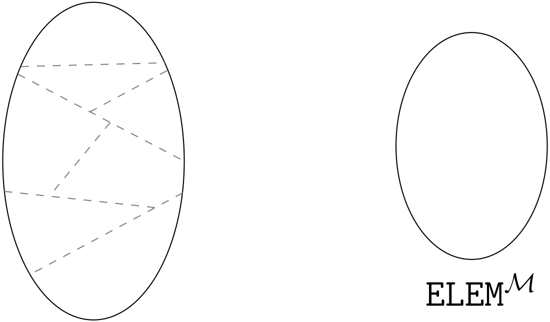

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$$\boxed{\text{STEP I. Guess the set of } \texttt{INDEX}\ \textit{types}}$$

- Consider the set K of all `INDEX` *atoms* in F (plus equalities with the $\mathbf{c}$ constants)
- Let $\{M_1, \ldots, M_q\}$ be the the set of *maximal* and *consistent* sets of literals built out of K
  - Each $L(x, \mathbf{c})$ in every M_h is an atom of K or its negation
  - All the M_h's are mutually exclusive
- Every element of $\texttt{INDEX}^{\mathcal{M}}$ has to realize a type M_h:

$$\mathcal{M}_I \models \forall x. \left(\bigvee_{j=1}^{q} \bigwedge_{L \in M_j} L(x, \mathbf{c}) \right)$$

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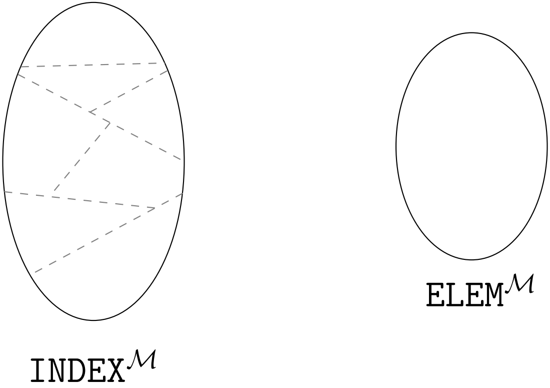

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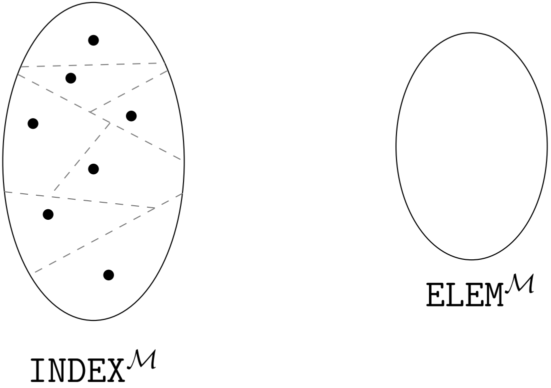

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1. Each b_h realizes the corresponding type

$$\mathcal{M}_I \models \bigwedge_{j=1}^{q} \bigwedge_{L \in M_j} L(b_j, \mathbf{c})$$

2. The instantiation

$$\bigwedge_{\sigma:\mathbf{i} \to \mathbf{b}} \psi(\ \mathbf{i}\sigma, a(\mathbf{i}\sigma), \mathbf{c}, a(\mathbf{c})\ )$$

is consistent

Decision Procedure for $\mathtt{ARR}^2(T_I, T_E)$

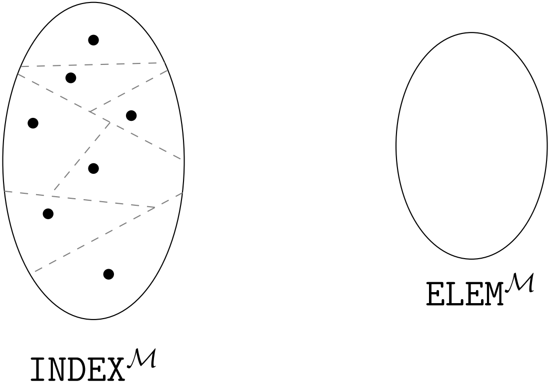
$$F := \exists \mathbf{c}\ \forall \mathbf{i}\ .\psi(\ \mathbf{i}, a(\mathbf{i}), \mathbf{c}, a(\mathbf{c})\ )$$

$$\rightsquigarrow$$

$$F_1 := \exists \mathbf{b}\, \exists \mathbf{c} \left[\begin{array}{l} \forall x. \left(\bigvee_{j=1}^{q} \bigwedge_{L \in M_j} L(x, \mathbf{c}) \right) \ \wedge \\ \bigwedge_{j=1}^{q} \ \bigwedge_{L \in M_j} L(b_j, \mathbf{c}) \ \wedge \\ \bigwedge_{\sigma : \mathbf{i} \to \mathbf{b}} \psi(\mathbf{i}\sigma, a(\mathbf{i}\sigma), \mathbf{c}, a(\mathbf{c})) \end{array} \right]$$

Decision Procedure for the multi-sorted case

STEP III. Substitute the tuple $a(\mathbf{b}) * a(\mathbf{c})$ with a tuple $\mathbf{e}$ of `ELEM` constants



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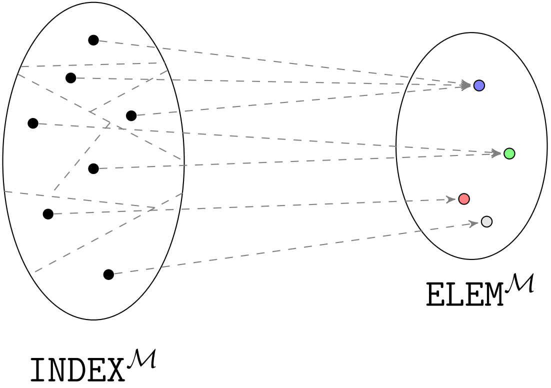

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$a(\mathbf{b}) * a(\mathbf{c}) \;\rightsquigarrow\; \mathbf{e}$

$$F_2 := \exists \mathbf{b}\, \exists \mathbf{c} \left[\begin{array}{l} \ldots \wedge \\ \bar{\psi}(\mathbf{b}, \mathbf{c}, \mathbf{e}) \wedge \\ \displaystyle\bigwedge_{d_m, d_n \in \mathbf{b}*\mathbf{c}} \; \bigwedge_{l=1}^{s} (d_m = d_n \to e_{l,m} = e_{l,n}) \end{array} \right]$$

functional consistency

Decision Procedure for the multi-sorted case

STEP IV. "Split" the formula F_2 in `INDEX` and `ELEM` parts

$$F_2 := \exists \mathbf{b}\, \exists \mathbf{c} \left[\begin{array}{l} \forall x. \left(\bigvee_{j=1}^{q} \bigwedge_{L \in M_j} L(x, \mathbf{c}) \right) \; \wedge \\ \bigwedge_{j=1}^{q} \quad \bigwedge_{L \in M_j} L(b_j, \mathbf{c}) \; \wedge \\ \bar{\psi}(\mathbf{b}, \mathbf{c}, \mathbf{e}) \;\; \wedge \quad \bigwedge_{d_m, d_n \in \mathbf{b} * \mathbf{c}} \quad \bigwedge_{l=1}^{s} (d_m = d_n \to e_{l,m} = e_{l,n}) \end{array} \right]$$

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$$F_I := \exists \mathbf{b}\, \exists \mathbf{c} \left[\begin{array}{l} \forall x. \left(\bigvee_{j=1}^{q} \bigwedge_{L \in M_j} L(x, \mathbf{c}) \right) \wedge \\ \bigwedge_{j=1}^{q} \bigwedge_{L \in M_j} L(b_j, \mathbf{c}) \wedge \\ \bar{\psi}(\mathbf{b}, \mathbf{c}) \end{array} \right]$$

$$F_E := \bar{\psi}(\mathbf{e})$$

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SAT assignment

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$\boxed{\textsc{Step V. Check if } F_I \text{ is } T_I\text{-sat and if } F_E \text{ is } T_E\text{-sat}}$

[1*] With divisibility predicates $\{D_k\}_{k \geq 2}$.

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$\Rightarrow$ $\exists^*\forall$-fragment

$\Rightarrow$ Quantifier-free fragment

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$\Rightarrow$ $\exists^*\forall$-fragment

- ✔ Difference Logic*
- ✔ Presburger*
- ✔ Presburger* + exp [Semënov84]
- ✔ Real Arithmetic

...

$\Rightarrow$ Quantifier-free fragment

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Application: deciding the safety of $\mathsf{simple}^0_{\mathcal{A}}$-*programs*

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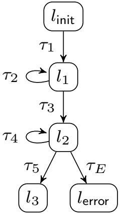
- Flat control-flow structure
- Every loop τ has a Flat Array Property as acceleration



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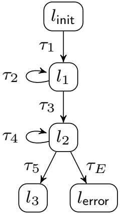

Theorem

The unbounded reachability problem for $\mathsf{simple}^0_{\mathcal{A}}$*-programs is decidable.*

Practical observations

- $\mathsf{simple}^0_{\mathcal{A}}$-*programs*:
    - initialization
    - copying
    - testing
    - swapping
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  - swapping
  - etc.

- The SMT-Solvers Z3 and CVC4 fail on (some) proof obligations
  - especially the satisfiable ones (derived by unsafe programs)

Conclusion

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   - Fully declarative
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Thank you! Questions?

References I

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